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1994 J. Phys. A: Math. Gen. 27 L317

(http://iopscience.iop.org/0305-4470/27/10/003)

J. Phys. A: Math. Gen. 27 (1994) L317–L321. Printed in the UK

LETTER TO THE EDITOR

High-order generators of Lie algebras and high-order central extensions

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Received 16 March 1994

Abstract. High-order generators of Lie algebras are defined and their basic properties are discussed in this letter. In 'classical' cases, the high-order generators are simply the enveloping algebras of Lie algebras, while, in 'quantum' cases, we can define the high-order central extensions of (infinite-dimensional) Lie algebras and can see that the high-order generators are no longer the enveloping algebras of Lie algebras. As an example, we calculate the first-high-order central extensions of Virasoro algebras and find that the first-order and usual-order central extensions are closely related to each other.

High-order generators of Lie algebras can be thought of as the enveloping algebras of the corresponding Lie algebras in the 'classical' case (the commutation relations between the generators have no central extensions); this is described in the following section. However, the study of high-order generators of Lie algebras proves to be useful, especially if we consider high-order central extensions of infinite-dimensional Lie algebras, such as Kac–Moody and Virasoro algebras. In such cases, the high-order generators of Lie algebras are no longer trivial. In [1], the high-order generators of Virasoro algebras were defined. In this letter we define the high-order generators of Lie algebras (finite and infinite dimensional) generally and calculate the first-order central extensions of Virasoro algebras as an example. We find that the usual-order central extensions and the first-order ones are closely related to each other. Since the central extensions have wide applications in physics, we hope that the high-order central extensions will also have some applications in physics.

Let G be a compact Lie group, with unit E, and $D(G)$ be a representation of G. The group parameter is denoted by α. The parameter of E is $\alpha = 0$. The parameter of an infinitesimal element of E is denoted by α_j, $j = 1, \ldots, N$ where N is the rank of the Lie algebra g. According to Taylor's theorem

$$D(A) = 1 + \sum_{j=1}^{N} \alpha_j \frac{\partial}{\partial \alpha_j} D(0) + \frac{1}{2} \sum_{i,j=1}^{N} \alpha_i \alpha_j \partial_i \partial_j D(0) + \cdots \tag{1}$$

where ∂_i denotes $\partial/\partial\alpha_i$. The generators I_j of Lie algebra g are defined by $I_j = \mathrm{i}\partial_j D(\alpha)|_{\alpha=0} = \mathrm{i}\partial_j D(0)$ (where $\mathrm{i}^2 = -1$) with the well known commutation relations [5]

$$[I_j, I_k] = \sum_l C_{jk}^l I_l \qquad \text{or} \qquad [\partial_j D(0), \partial_k D(0)] = \sum_l C_{jk}^l \partial_l D(0) \tag{2}$$

where C_{jk}^{l} are the structure constants of g. From (2), one may ask what the commutation relations of $[\partial_a\partial_b D(0), \partial_c D(0)]$ are. In this section we try to answer this question.

Let R, S and $T \in G$, with the group parameters denoted by r, s and t, respectively. Let $RS = T$, $t_j = f_j(r; s)$ where f_j are the combinational functions. Introduce

$$I_j = \mathrm{i}\partial_j D(0) \qquad I_{jk} = \mathrm{i}^2\partial_j\partial_k D(0) \qquad I_{jkl} = \mathrm{i}^3\partial_j\partial_k\partial_l D(0), \ldots \tag{3}$$

and

$$\begin{aligned} S_{jk}(r) &= \frac{\partial}{\partial r_k} f_j(r; s)|s = \bar{r} \\ T_{lk}^{j}(r) &= \frac{\partial^2}{\partial r_k \partial r_l} f_j(r; s)|s = \bar{r} \\ R_{lkm}^{j} &= \frac{\partial^3}{\partial r_l \partial r_k \partial r_m} f_j(r; s)|s = \bar{r}, \ldots \end{aligned} \tag{4}$$

where $\bar{r}$ is the group parameter of R^{-1}. Since I_j are the generators of the Lie algebra g, we call I_{jl} the second-order generators of g and I_{jkl} the third-order generators of g, etc. The values of $S_{jk}(r)$, $T_{lk}^{j}(r)$ and $R_{lkm}^{j}(r)$ in (4) at the point $r = 0$ (denoted by S_{jk}, T_{lk}^{j} and R_{lkm}^{j}) play the 'structure constant' roles which are similar to that of C_{jk}^{i}, the structure constant of Lie algebra g. One can prove that

$$S_{jk} = \delta_{jk} \qquad T_{lk}^{j} = R_{lkm}^{j} = \cdots = 0. \tag{5}$$

Furthermore, one can calculate the commutation relations of high-order generators such as $[I_k, I_{mn}]$, $[I_{fn}, I_{km}]$. The calculation is similar to that of $[I_m, I_n]$ but it is more tedious. Of course, these commutation relations are 'trivial' if we do not consider their central extensions. Some of our results are

$$[I_k, I_{mn}] = \sum_j (I_j I_n \partial_m S_{jk} - I_{jn}\partial_k S_{jm}) + \sum_j (I_j I_m \partial_n S_{jk} - I_{jm}\partial_k S_{jn}) + \mathrm{i}\sum_j (\partial_n\partial_m S_{jk} - \partial_k T_{mn}^{j}) \tag{6}$$

$$\begin{aligned} [I_{fn}, I_{km}] = {} & -i\sum_j (\partial_f\partial_n T_{km}^{j} - \partial_k\partial_m T_{fn}^{j}) - \sum_j I_{jm}\partial_f\partial_n S_{jk} + \sum_j I_{jn}\partial_k\partial_m S_{jf} - \sum_j I_{jk}\partial_f\partial_n S_{jn} \\ & - \sum_{jl} (\partial_n S_{jk}\partial_f S_{lm} - \partial_m S_{jf}\partial_k S_{ln}) - \sum_{jl} (\partial_f S_{jk}\partial_n S_{lm} - \partial_k S_{jf}\partial_m S_{ln}) \\ & + I_{nkm} I_f - I_{mfn} I_k + I_{fkm} I_n - I_{fkn} I_m - I_n I_{km} I_f + I_m I_{fn} I_k - I_f I_{km} I_n + I_k I_{fn} I_m \\ & \text{etc.} \end{aligned} \tag{7}$$

Remember that $\partial_k S_{jm}$, $\partial_f\partial_n T_{km}^{j}$, etc, take values at the point $r = 0$. It is interesting to verify the following identity:

$$[I_k, I_{mn}] + [I_m, I_{nk}] + [I_n, I_{km}] = 0 \tag{8}$$

or

$$[I_k, I_{mn}] + (k \to m \to n \text{ cyclic permutation terms}) = 0. \tag{9}$$

Furthermore, we have the following conclusion:

$$[I_a, I_{bc\ldots d}] + (a \to b \to c \to \cdots \to d \text{ cyclic permutation terms}) = 0. \quad (10)$$

Note that (9) and (10) are due to the fact that high-order generators are simply the enveloping algebras of I_a in classical cases (without central extensions in the commutation relations). However, in a quantum theory, (9) and (10) may no longer be correct (see the next section).

Central extensions of Lie algebras have found many applications in conformal-field theory and other theories. Of course, the formulae (6)–(9) give no more information because the high-order generators of g are only the enveloping algebras of I_k, the generators of g. In this section we will discuss the high-order central extensions of the high-order generators. They cannot simply be considered as the enveloping algebras of I_k in this case. In classical theory, without central extensions, the generators I_k of Lie algebra g satisfy (2), namely, they satisfy the Poisson bracket relations

$$[I_a, I_b]_{\mathrm{PB}} = \sum_d C^d_{ab} I_d. \quad (11)$$

In a quantum theory, following Dirac's quantization procedure, the corresponding quantum commutation relations are

$$[I_a, I_b] = \mathrm{i}\hbar \sum_d C^d_{ab} I_d + \mathrm{O}(\hbar^2) \quad (12)$$

where $\hbar$ is the Plank constant. We recognize that the unspecified terms of order $\hbar^2$ are the usual central extension terms. For simplicity, these terms can be supposed to consist of c numbers which can be derived by Jacobi identities. For compact semi-simple finite-dimensional Lie algebras (2), the central extensions are essentially trivial [2]. In order to obtain a non-trivial theory, we should consider infinite-dimensional algebras such as Kac–Moody and Virasoro algebras.

Now we consider the central extensions of high-order generators of Lie algebra g (a finite- or infinite-dimensional Lie algebra). In quantum theory, following Dirac's quantization procedure, (6) can be rewritten as

$$\begin{aligned}[I_k, I_{mn}] = {} & \mathrm{i}\hbar \sum_j (I_j I_n \partial_m S_{jk} - I_{jn} \partial_k S_{jm}) + \mathrm{i}\hbar \sum_j (I_j I_m \partial_n S_{jk} - I_{jm} \partial_k S_{jn}) \\ & + (\mathrm{i}\hbar)\mathrm{i} \sum_j (\partial_n \partial_m S_{jk} - \partial_k T^j_{mn} + \mathrm{O}(\hbar^2) + \mathrm{O}(\hbar^3). \end{aligned} \quad (13)$$

The unspecified terms of order $\mathrm{O}(\hbar^3)$ can be assumed to be c numbers which can be derived by Jacobi identities. Moreover, the unspecified terms of order $\mathrm{O}(\hbar^2)$ may not be zero (these terms are assumed to be zero in [1]) and can no longer be assumed to be c numbers. We call the terms of order $\mathrm{O}(\hbar^3)$ and $\mathrm{O}(\hbar^2)$ the usual-order and the first-high-order central extensions, respectively. One can define the second-, third-, ..., high-order central extensions similarly. It is important to point out that the high-order central extensions of a finite Lie algebra are trivial by a suitable transformation and, hence, can also be set to be zero. In this section, we will calculate the first-high-order central extensions of the Virasoro algebra as an example.

High-order generators of the Virasoro algebra were introduced in [1] and can be regarded as the enveloping algebras of the Virasoro algebra if the central extensions are zero. The high-order generators of Virasoro algebra are defined by [1]

$$L_m^k = (-)^k z^{m+k} \frac{\mathrm{d}^k}{\mathrm{d}z^k} \qquad k = 1, 2, \ldots \qquad m = 0, \pm 1, \pm 2, \ldots \tag{14}$$

which hold the following commutation relations

$$[L_m^k, L_n^l] = \left(\sum_{p=1}^{k} (-)^p C_p^k B_p^{n+l} - \sum_{p=1}^{l} (-)^p C_p^l B_p^{m+k} \right) L_{m+n}^{k+l-p} \tag{15}$$

where

$$C_n^m = \frac{m!}{(m-n)!n!} \qquad B_n^m = \begin{cases} \dfrac{m!}{(m-n)!} & n \leqslant m \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

The usual-order central extensions of (15), namely the unspecified terms of order $O(\hbar^3)$ in (13) were also calculated in [1]. For example

$$[L_m^1, L_n^2] = (2m-n)L_{m+n}^2 - m(m+1)L_{m+n} - \frac{c}{24} m(m^2-1)(m-2)\delta_{m+n,0} \tag{17}$$

$(L_m^1 = L_m)$ where c is the constant number appearing in the commutation relations of Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0}. \tag{18}$$

However, according to (13), (17) does not contain the terms of order $O(\hbar^2)$. Since $O(\hbar^2)$ cannot be c-numbers, they must be the combination of L_m. So, the first-high-order central extensions can be regarded as being the modifications of the terms $m(m+1)L_{m+n}$ appearing in (17). The most general form of (17), including the usual-order and first-order central extensions, can be written as

$$[L_m^1, L_n^2] = (2m-n)L_{m+n}^2 - m(m+1)L_{m+n} + \sum_j R_{mn}^j L_j + Q(m,n) \tag{19}$$

where R_{mn}^j and $Q(m,n)$ are c numbers. We make the following ansatz:

$$R_{mn}^j = T(m,n)\delta_{m+n,j} \qquad T(m,n) = \delta_{m+n,0} \frac{c}{12}(am^2 + bm + c) \tag{20}$$

and

$$Q(m,n) = \delta_{m+n,0} \frac{c}{12}(Am^4 + Bm^3 + Cm^2 + Dm + E) \tag{21}$$

where $a, \ldots, c, A, \ldots, E$ are constants. Using (20) and (21), following the Jacobi identities

$$[L_l, [L_m, L_n^2]] + (l \to m \to n \text{ cyclic permutation terms}) = 0$$

one obtains the following solution

$$a = 2A + 1 \qquad b = 1 - B \qquad c = 0 \qquad a = 1 - 2C \qquad E = 0 \qquad D \in R. \quad (22)$$

Written explicitly, (19) is

$$[L_m^1, L_n^2] = (2m - n)L_{m+n}^2 - m(m+1)L_{m+n} + \frac{c}{12}((4A+2)m^2 + (2-2B)m)L_{m+n}$$
$$+ \frac{c}{12}(Am^4 + Bm^3 - Am^2 + Dm)\delta_{m+n,0}. \quad (23)$$

Equation (23) shows that the usual-order and first-order central extensions are closely related to each other. One may assume that all the central extensions, including the usual-order ones, are related to each other. Notice that if we choose $A = -C = -\frac{1}{2}$ and $B = 1$, (23) coincides with (17). Also, if we choose $A = B = C = 0$ then (23) can be written as

$$[L_m^1, L_n^2] = (2m - n)L_{m+n}^2 - m(m+1)L_{m+n} + \frac{c}{24}(m^2 + m)L_{m+n}. \quad (24)$$

There are no usual-order central extensions in (24).

In classical theory, we have (9) and (10); however, in quantum theory, (9) and (10) are no longer correct. Note that (9) and (10) are only correct because of the fact that in the classical theory, the high-order generators are simply the enveloping algebras of Lie algebra g. For example, we have the following relations

$$I_{mn} = I_m I_n + \mathrm{i} I_l \partial_n S_{lm}. \quad (25)$$

This suggests that in the quantum theory we cannot simply treat the high-order generators as the enveloping algebras of g. Remember that (23) is the result of the Jacobi identity. Notice that if we set the first-order central extensions to be equal to zero (then (23) goes back to (17)) then the commutation relations in (17) satisfy (9) and (10). Since there are three free parameters in (23) and no free parameters in (17) besides c, we see that we have to lose the three free parameters in order to make (23) satisfy (9) and (10).

We have calculated the first-high-order central extensions of Virasoro algebras as an example. One can calculate other high-order central extensions between the high-order generators of Virasoro algebras similarily using Jacobi identities. Also, one can consider the high-order central extension of Kac–Moody algebras [6]. Since we have defined the high-order generators of Lie algebras, one can define the high-order generators of Kac–Moody algebras. For example, one can define $T_n^{ab} = I_{ab}z^n$ as the first-high-order generators of Kac–Moody algebras and using this definition can calculate the first-high-order central extensions. Unfortunately, the calculation is more complicated than that of Virasoro algebras. We anticipate that the high-order central extensions will play a role in some branches of physics, though the geometric or physical explanation of the high-order central extensions of infinite-dimensional algebras is obscure.

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